Chaos-induced synchronization in discrete-time oscillators driven by a random input

Ali A. Minai and Tirunelveli Anand
Department of Electrical & Computer Engineering and Computer Science, University of Cincinnati, Cincinnati, Ohio 45221-0030
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The research reported in this paper focuses on the response of identical discrete-time neural oscillators to a random telegraph signal input. It is shown that there are well-defined domains in which the random input can synchronize even a large population of oscillators within a few hundred steps. The presence of chaos is shown to be essential for the synchronization, which suggests a possible role for chaos in spatially extended physical systems. The effect of independent noise on the system is also studied. [S1063-651X(98)05102-2]

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Recently there has been considerable interest in the synchronization of chaotic oscillators [1–5]. While most of the work has focused on coupled continuous-time oscillators, the possibility of synchronizing chaotic maps via a driving signal also has been investigated. In particular, the issue of using a common ''noiselike'' input to synchronize a group of identical oscillators has generated some discussion [6–10]. It was suggested initially [6] that identical quadratic maps driven by a common noise but starting from different initial states would synchronize spontaneously over a period of about 10^6 steps. Subsequently, it was pointed out [7,9] that this apparent synchronization was the result of finite precision in representing the system state and that the time to synchronization scaled exponentially with increasing precision, rendering this method impractical. Recently we have reported that certain types of nonlinear maps can indeed be synchronized reliably and rapidly by a noiselike input [11]. In this paper we look in closer detail at one particularly simple type of driving input: a random telegraph signal. Our results are closely related to previous work on random maps [12–14], but focus more explicitly on the role of chaos in producing synchronization.

The map we consider is based on a neural oscillator model proposed by Wang [15,16] and is a discrete-time version of the Wilson-Cowan oscillator [17]. The map is given by

$$z_{i+1} \equiv F(z_t, u_t) = \tanh[\mu(a z_t + u_t)] - \tanh[\mu b z_t], \quad (1)$$

where a, b, and μ are parameters and u_t is the drive. Note that the drive is added *inside* the nonlinearity and that, unlike the quadratic map, the dynamics of F are defined and bounded over the whole real line. The system described by Eq. (1) will be called a μ-a-b oscillator. It can be shown that if $a \geqslant 2b$ and $u_t = 0\ \forall t$, the map undergoes period-doubling bifurcation to chaos as μ is increased [15,16]. When the origin is unstable, the dynamics of z_t has two basins of attraction separated by $z_t = 0$. If the initial condition z_0 and u_t have the same sign for all t, the dynamics remains confined to the basin with the corresponding sign.

In the fixed input case $u_t = u\ \forall t$ if μ is set such that the map is chaotic for $u = 0$, increasing $|u|$ causes a series of period-halving bifurcations in both basins, leading ultimately to cycles of period 2 for large fixed drives. There is also a hysteretic effect at the basin boundary, so that the sign of the state for very small drive values is determined in some cases by the sign of the initial condition [18]. Figure 1 shows the bifurcation diagram and the Lyapunov exponents of the map for fixed input $-0.25 \leqslant u \leqslant 0.25$, starting from a positive initial condition. They are calculated numerically from

$$\lambda_F(u) = \frac{1}{T+1} \sum_{t=\tau}^{T+\tau} \ln|F'(z_t)|$$
$$= \frac{1}{T+1} \sum_{t=\tau}^{T+\tau} \ln|\mu a\ \mathrm{sech}^2[\mu(a z_t + u)] - \mu b\ \mathrm{sech}^2[\mu b z_t]|, \quad (2)$$

where τ is the transient length and T is taken to be very large.

If u_t is time varying, the system effectively jumps between different dynamic regimes, each with its own Lyapunov exponent. Previous work [12–14] has shown that

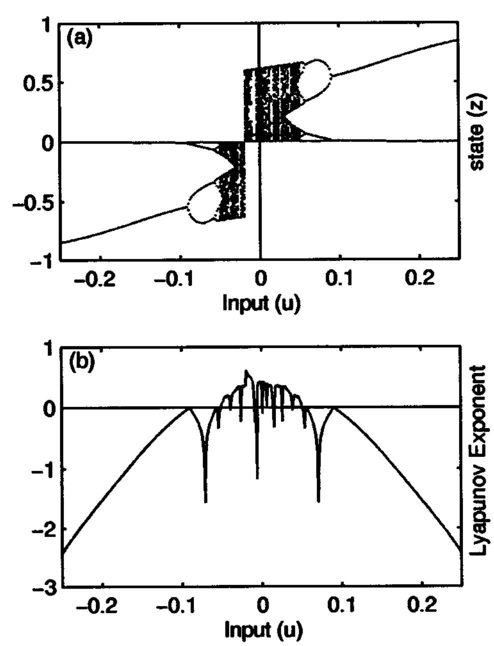

FIG. 1. (a) Bifurcation diagram for a 5-5-1 oscillator as input u is changed. (b) Numerically calculated Lyapunov exponent for the oscillator at different input values. The initial condition is positive in all cases.

if a population of identical nonlinear maps starts from slightly different initial conditions and is subject to a common noise, it will collapse into a small number of phase-locked clusters, and possibly a single synchronized group, if the Lyapunov exponent of the noisy system is negative. Clearly, however, a negative Lyapunov exponent is not sufficient to guarantee global synchronization, only collapse to zero volume in phase space. The key requirement,in addition to negative Lyapunov exponents, is to provide a mechanism that precludes the formation of multiple clusters. Chaos provides such a mechanism.

Consider a population of identical maps $F(z^i,u^i)$, which receives a random telegraph signal $u_t \in \{\alpha,\beta\}$, $0<\alpha<\beta$, as common input. The values α and β are chosen such that $u_t=\alpha$ puts the map in the chaotic regime while $u_t=\beta$ puts it in the k-cycle regime. The oscillators start with different positive initial conditions and in the chaotic regime, so that the probability of accidental synchronization is zero at infinite precision. When u_t switches to β, the system has k domains of attraction, one for each phase of the k cycle, and the trajectories collapse towards $m \leqslant k$ small areas of phase space. If u_t remains at β long enough, the collapse is complete and the population divides into m phase-locked clusters. If u_t now changes to α, the clusters remain phase locked, but the intercluster phase relationships are destroyed by chaos. When u_t changes back to β, each cluster chooses one of the k available phases in the periodic regime, leading to $n \leqslant m$ clusters. Clearly, each time a β-α-β switch occurs, the number of clusters cannot increase, but there is a nonzero probability of decrease. If k is small (e.g., 2 or 4), the number of clusters goes rapidly to 1 with very high probability.

Two conditions are crucial to the success of the scenario described above: (i) enough time spent in the periodic regime to allow sufficient collapse of trajectories (i.e., globally negative Lyapunov exponent) and (ii) occasional reversion to the chaotic regime to disrupt intercluster phase relationships (i.e., intermittently positive Lyapunov exponent). Condition (i) is of course essential to achieve the necessary convergence of trajectories. Condition (ii) is more subtle. The reversion to chaos essentially makes the whole phase space (and thus all k domains of attraction) *independently available* to each cluster without disrupting the phase locking among its members, thus allowing the clusters to merge without the possibility of splitting. Reversion to just a cycle of period $2^i k > k$ does not produce synchronization because, while it does increase the number of phase-space points available to each cluster, it does not disrupt the *relative* phase relationships between the clusters with respect to the domains of attraction in the k-cycle regime. Similarly, it is also important that the input signal to the system be random to ensure a broad sampling of intercluster phase combinations. A periodic square-wave input can easily produce a phase locking between the cluster dynamics and the input, trapping the system in a multicluster situation.

We use a population of 5-5-1 oscillators to empirically explore the phenomena described above. The input signal is given by

$$u_t=\begin{cases}\alpha & \text{with probability } p\\ \beta & \text{with probability } q=1-p.\end{cases} \tag{3}$$

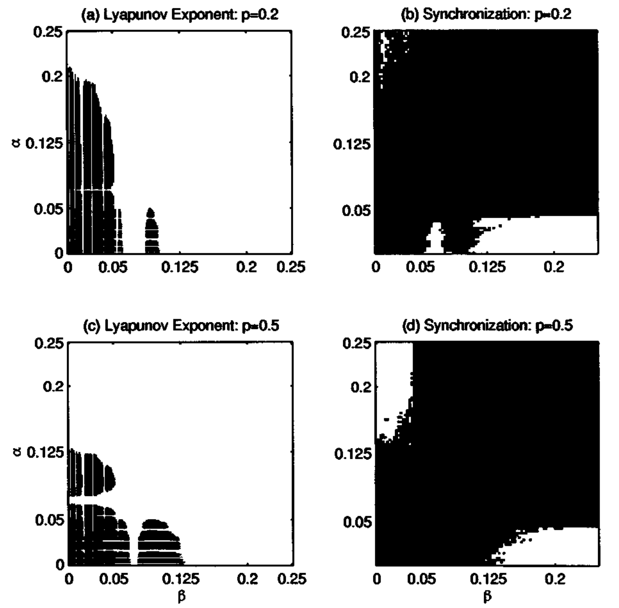

FIG. 2. Domains of negative and positive Lyapunov exponents Λ_F for a 5-5-1 oscillator with (a) $p=0.2$ and (c) $p=0.5$. The white area indicates L^- and the black area L^+. (b) and (c) Actual domains of synchronization (white areas) for a population of fifty 5-5-1 oscillators. Each point in the α-β plane represents one simulation of 1000 steps. Longer simulations increase the domains of synchronization slightly, indicating slow synchronization near the L^+-L^- boundary.

The Lyapunov exponent of the driven system can be calculated by substituting u_t for u in Eq. (2). However, we approximate its mean value by

$$\Lambda_F = p\lambda_F(\alpha) + q\lambda_F(\beta). \tag{4}$$

Clearly, once α and β are fixed, the possibility of synchronization is controlled wholly by the p parameter. If p is small, the system spends long intervals of time in the periodic regime and synchronization occurs as described above. However, if p is too small, not enough time is spent in the chaotic regime and synchronization is slow or absent altogether. If p is relatively high, the time spent in the periodic regime may be too short to allow for complete collapse to a cycle without interruption by chaos. However, if there is sufficient convergence during each periodic phase (i.e., Λ_F is negative), this will accumulate over time to produce synchronization, albeit slowly.

Consider the case when $p=0.2$. Figure 2(a) shows the domains corresponding to the negative and positive Lyapunov exponents of the system in α-β space. Let the white region (with negative Λ_F) be denoted L^- and the black (with positive Λ_F) L^+. Following Yu, Ott, and Chen [13], we expect that if $\alpha,\beta \in L^-$, all trajectories will collapse to a small set of clusters. The domain of synchronization, however, is only that subset of L^- where the two conditions given above are satisfied. Based on the bifurcation diagram of the oscillator (Fig. 1), the transition between the chaotic and periodic regimes occurs at $u^* \approx 0.05$. Condition (ii) (occasional occurrence of chaos) is therefore satisfied in the regions $R_1 \equiv \{\alpha<u^*,\ \beta>u^*\}$ and $R_2 \equiv \{\alpha>u^*,\ \beta<u^*\}$.

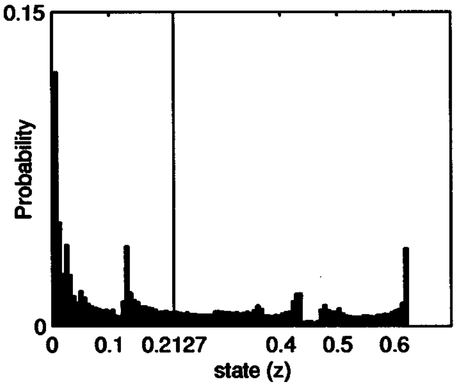

FIG. 3. Invariant distribution of a 5-5-1 oscillator with $u=0.01$. The distribution is calculated numerically, using 10^5 data points. The vertical line at 0.2127 indicates the basin boundary for the two phases of the period-2 attractor obtained when u is larger.

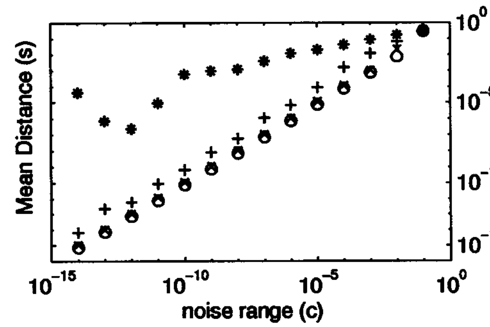

FIG. 4. Mean separation of trajectories $\langle s \rangle_t$ in five 5-5-1 oscillators with independent uniform noise between $\pm c$ added to each oscillator's driving signal. The average was calculated over 5×10^4 steps following 1000 steps for synchronization. The values for p are 0.01 (○), 0.05 (×), 0.2 (+), and 0.5 (*). Note the linear dependence of $\langle s \rangle_t$ on c for $p \leqslant 0.2$.

Condition (i) is satisfied in region L^-. Thus the theoretical domain of synchronization is given by $S \equiv L^- \cap \{R_1 \cup R_2\}$. However, close to the boundary between L^- and L^+, synchronization may take a long time because Λ_F is only slightly negative. Based on this analysis, synchronization is expected to occur in two bands of R_1 and for $\alpha > 0.2$ in R_2. Figure 2(b) shows the results for a population of fifty 5-5-1 oscillators in the α-β plane and the results are indeed as expected. Figures 2(c) and 2(d) show the results for $p = 0.5$, which are symmetric in the α-β plane, as expected from Eq. (3). Note that, in both cases, each simulation was run only for 1000 time steps, so the broad coverage of the theoretical synchronization domain S attests also to the speed of synchronization.

The synchronization process can be appreciated easily by considering the case where β puts the system in the period-2 regime. The basins of attraction for the two phases are divided by the fixed point z^* of the $F(z,u)$ map. For all but the smallest u values, $\tanh[\mu(az^*+u)] \approx 1$ if μ and a are large enough, so in the period-2 regime, z^* is given approximately by the implicit equation $z^* = 1 - \tanh[\mu b z^*]$ and is independent of u. For the 5-5-1 map, $z^* \approx 0.2127$ for all but the smallest values of u. Figure 3 shows the invariant distribution of the 5-5-1 map at $u=0.01$ with the z^* boundary indicated. It is clear that if $\alpha = 0.01$ and β is in the period-2 regime, both basins of attraction are explored in the chaotic regime (with probability 0.553 11 for phase 1 and 0.446 89 for phase 2), thus leading to synchronization.

A key concern from the applied standpoint is the effect of independent noise on the synchronization mechanism. To test this, we added small amounts of independent noise η_t^i uniformly distributed between $\pm c$, to the drive for each oscillator i. The drive for i thus became $u_t^i = u_t + \eta_t^i$. We then tracked the mean absolute distance between trajectories $s_t \equiv 2[N(N-1)]^{-1} \Sigma_{i=1}^{N-1} \Sigma_{j=i+1}^{N} |z_t^i - z_t^j|$ after an initial synchronization period of τ time steps. Figure 4 plots the time average $\langle s_t \rangle_t$ against noise spread c, showing a strongly linear dependence for $p \leqslant 0.2$. For larger p, the time spent in the chaotic regime begins to magnify the noise more dramatically. Note, however, that even for p as high as 0.5, relatively large amounts of noise can be tolerated if synchronization is desired only to a low precision, which will often be the case in applications such as neural modeling. In this context, the added noise has another useful consequence: When the common part of u_t^i is switched off, the oscillators desynchronize spontaneously since they are in the chaotic regime near $u_t = 0$. Thus, adding a small amount of independent noise provides a desynchronization mechanism without disrupting synchronization, a fact of great significance for possible information processing applications [5].

To conclude, the main result of the research described in this paper is that chaos can help nonlinear oscillators synchronize in response to a common random driving input. This synchronization would not happen generically in the absence of chaos (except in the trivial case of convergence to a fixed point), which points to a potentially useful role for chaotic regimes in physical systems [19]. It is also intriguing that two relatively disordered behaviors (chaos and noiselike input), in conjunction, facilitate the emergence of a highly organized behavior: synchronization.

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